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Introducing transverse vertices into the gauge technique

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Abstract. We present a new algorithm for extending the gauge technique beyond the simplest version. The algorithm introduces transverse corrections in an intrinsically non-perturbative way and represents a significant improvement on previously advocated methods. For scalar electrodynamics the resulting gauge technique equation is analysed in the infrared and ultraviolet limits.

1. Introduction

The gauge technique (GT) (Salam 1963, Delbourgo and West 1977a, Delbourgo 1979) is a non-perturbative self-consistent approximation method for calculating gauge field theory Green functions. It can be applied to any gauge theory (and some non-gauge theories) and has the virtue of satisfying the gauge identities at every stage of the calculation. The simplest (or 'zeroth-order') version of the GT combines the gauge identities, Källén–Lehmann spectral representation and the Schwinger–Dyson equation for the source two-point function. These ingredients suffice to determine the purely longitudinal components, G^{L}, of the photon amputated, connected amplitudes, G (Delbourgo and West 1977a), and the *exact* infrared (IR) behaviour of the source two-point functions in scalar (SED) and spinor (QED) electrodynamics (Delbourgo and West 1977b), and vector electrodynamics (Delbourgo 1977). In each case the results concur with those obtained by other methods (Gorkov 1956, Ball *et al* 1978, Hagen 1963, Kibble 1968). Further, the ultraviolet (UV) behaviour of the electron propagator agrees with that found by Baker and Johnson (Khare and Kumar 1978). Unfortunately the zeroth-order GT (in general) only respects gauge covariance in these asymptotic limits (Slim 1981a, Delbourgo *et al* 1981), due to the neglect of transverse vertices in the analyses. This omission also means that the GT (in lowest order) cannot provide a suitable explanation for every phenomenon associated with a gauge theory. For example, the anomalous magnetic moment of the electron in QED is associated with the transverse vertex $\sigma_{\mu\nu}k^{\nu}$. The neglect of transverse contributions is the main criticism that can be levelled at the simplest version of the GT.

There have been several attempts to extend the GT beyond the lowest order. The structure of the *full* transverse contribution has been found in two-dimensional space–time (Gardner 1981, Delbourgo and Thompson 1982), enabling exact representations for $G_{\mu} = S\Gamma_{\mu}S$, $G_{\mu\nu} = S\Gamma_{\mu\nu}S$, etc. Delbourgo and co-workers have suggested that the form of the transverse contribution in four dimensions could be obtained by analysing the Schwinger–Dyson equation for the full three-point amplitude G_{μ}. The resulting solution would be used (instead of G^{L}_{μ}) in a revised GT analysis of the equation for the two-point function, leading to a refined approximation of the true propagator. It is implicit in this approach that a spectral ansatz for $G_{\mu\nu}$ is available; however, spectral representations for the full $G_{\mu_1\ldots\mu_n}$ with $n>1$ are unknown at present and hence this method is not viable. Slim (1981b) has advocated a simpler version of the idea based on the spectral representation for G^{L}. It amounts to an amalgamation of the GT with perturbation theory (PT) and only employs longitudinal amplitudes. An approximate transverse vertex for QED, which yields the correct IR and UV behaviours of the electron propagator, has recently been obtained (King 1983). The analysis relies solely on PT, begins with an *ad hoc* modification of the $\mathrm{O}(e^2)$ transverse vertex, and is only valid in the IR and UV limits. King stresses the need to introduce tranverse vertices into the GT in order to treat properly the problem of overlapping divergences.

In § 2 we derive a new, non-perturbative, approximation to the full three-point transverse amplitude, G^{T}_{μ}. Because G_μ, in QED, is a linear combination of eight transverse covariants and four longitudinal covariants, we choose to work with SED where G_μ consists of one longitudinal term and one transverse term. Section 3 describes the introduction of the new expression into the GT and results in a refined GT equation for the spectral function. The IR and UV properties of this equation are discussed in § 4. Section 5 summarises the article and discusses some topics for future research.

2. The transverse three-point amplitude G^{T}_{μ}

We begin by truncating the Schwinger–Dyson equation for G_μ (in SED) in a manner consistent with $\mathrm{O}(e^2)$ PT. Neglecting any terms that start at $\mathrm{O}(e^4)$ or higher implies that the zeroth-order GT ansatz (which consists of replacing the full $G_{\mu_1\ldots\mu_n}$ by $G^{\mathrm{L}}_{\mu_1\ldots\mu_n}$) can be employed on the right-hand side (RHS)† to obtain

$$
\begin{aligned}
&(p'^2-m_0^2)G_\mu(p',p)\\
&\quad=(p'+p)_\mu\Delta(p)-2\mathrm{i}e^2\int \bar{\mathrm{d}}^4t\, g_{\mu\sigma}D_0^{\sigma\pi}(t)G^{\mathrm{L}}_{\pi}(p-t,p)\\
&\qquad-\mathrm{i}e^2\int \bar{\mathrm{d}}^4t\,(2p'-t)_\lambda D_0^{\lambda\sigma}(t)G^{\mathrm{L}}_{\sigma\mu}(p'-t,t;p,k),
\end{aligned}
\tag{1}
$$

where from here on $k=p'-p$, Δ denotes the meson propagator, and we replace the full photon propagator by the undressed version D_0 as is usual in the GT. Adding (1) to the corresponding equation for $G_\mu(-p,-p')$ and employing the charge conjugation properties

$$
G_\mu(p',p)=-G_\mu(-p,-p'),\qquad G_{\mu\nu}(p',q';p,q)=G_{\mu\nu}(-p,q';-p',q)
$$

results in

$$
\begin{aligned}
&(p'^2-p^2)G_\mu(p',p)\\
&\quad=(p'+p)_\mu k^\nu G^{\mathrm{L}}_\nu(p',p)+2\mathrm{i}e^2\int \bar{\mathrm{d}}^4t\, g_{\mu\sigma}D_0^{\sigma\pi}(t)(k+t)^\nu\\
&\qquad\times G^{\mathrm{L}}_{\pi\nu}(p',t;p,k+t)+\mathrm{i}e^2\int \bar{\mathrm{d}}^4t\, D_0^{\lambda\sigma}(t)[(2p+t)_\lambda G^{\mathrm{L}}_{\sigma\mu}(p',t;p+t,k)\\
&\qquad-(2p'-t)_\lambda G^{\mathrm{L}}_{\sigma\mu}(p'-t,t;p,k)],
\end{aligned}
\tag{2}
$$

† $G^{\mathrm{L}}_{\mu_1\ldots\mu_n}$ is given by a spectral weighting over the Born terms contributing to $G_{\mu_1\ldots\mu_n}$. Since the ansatz contains lowest-order PT, any corrections to it must start at least at $\mathrm{O}(e^2)$.

where

$$G^{\mathrm{L}}_{\mu}(p',p)=\int\frac{\mathrm{d}\,W^2\,\rho(W^2)}{(p^2-W^2)(p'^2-W^2)}(p'+p)_{\mu}, \tag{3}$$

$$\begin{aligned}G^{\mathrm{L}}_{\mu\nu}(p',q';p,q)&\\ =\int\frac{\mathrm{d}\,W^2\,\rho(W^2)}{(p^2-W^2)(p'^2-W^2)}&\left(2g_{\mu\nu}-\frac{(2p'+q')_{\mu}(2p+q)_{\nu}}{[(p+q)^2-W^2]}\right.\\ &\left.-\frac{(2p'-q)_{\nu}(2p-q')_{\mu}}{[(p-q')^2-W^2]}\right),\end{aligned} \tag{4}$$

and ρ denotes the spectral function of the scalar meson. We will determine G^{T}_{μ} by projecting it out of (2).

Introducing the 'pseudo' projection operators† $L_{\mu\nu}$ and $T_{\mu\nu}$ given by

$$L_{\mu\nu}(p',p,k)\equiv k_{\mu}(p'+p)_{\nu}/(p'^2-p^2), \qquad T_{\mu\nu}(p',p,k)\equiv g_{\mu\nu}-L_{\mu\nu}(p',p,k), \tag{5}$$

we define the longitudinal and transverse parts of G_{μ} with respect to L and T by

$$G^{l}_{\nu}(p',p)\equiv L_{\mu\nu}(p',p,k)G^{\mu}(p',p), \qquad G^{\mathrm{t}}_{\nu}(p',p)\equiv T_{\mu\nu}(p',p,k)G^{\mu}(p',p). \tag{6}$$

Thus, from (3),

$$[G^{\nu\mathrm{L}}(p',p)]^{\mathrm{t}}=T^{\mu\nu}G^{\mathrm{L}}_{\mu}(p',p)=0,$$

and of course

$$[G^{\nu l}(p',p)]^{\mathrm{t}}=T^{\mu\nu}G^{l}_{\mu}(p',p)=0.$$

This knowledge, together with the definition

$$G^{\mathrm{T}}_{\mu}\equiv G_{\mu}-G^{\mathrm{L}}_{\mu},$$

then leads to the relations

$$G^{\mathrm{L}}_{\mu}(p',p)=G^{l}_{\mu}(p',p)+(p'+p)_{\mu}F(p',p), \tag{7a}$$

$$G^{\mathrm{T}}_{\mu}(p',p)=G^{\mathrm{t}}_{\mu}(p',p)-(p'+p)_{\mu}F(p',p), \tag{7b}$$

for some F. However, taking the divergence of $(7a)$ and noting that both G^{L}_{μ} and G^{l}_{μ} satisfy the gauge identity

$$k^{\mu}G_{\mu}(p',p)=\Delta(p)-\Delta(p')$$

yields

$$F(p',p)=0.$$

Applying $T_{\mu\nu}$ to (2) results in

$$\begin{aligned}G^{\mathrm{T}}_{\nu}(p',p)=2\mathrm{i}e^2T^{\mu}{}_{\nu}(p',p,k)&\int\frac{\mathrm{d}\,W^2\,\rho(W^2)}{(p^2-W^2)(p'^2-W^2)}\int\bar{\mathrm{d}}^4t\,D_0^{\sigma\pi}(t)\\ \times\left[\frac{t_{\mu}(2p+t)_{\sigma}(2p'+t)_{\pi}}{[(p+t)^2-W^2][(p'+t)^2-W^2]}\right.&\end{aligned}$$

† Projection operators are usually considered to be symmetric under interchange of indices. However, it is easy to verify that the non-symmetric L and T satisfy the usual properties of projection operators: $L^2=L$, $T^2=T$, $LT=TL=0$, $L+T=1$.

$$-g_{\mu\sigma}\left(\frac{(2p'+t)_\pi}{[(p'+t)^2-W^2]}+\frac{(2p+t)_\pi}{[(p+t)^2-W^2]}\right)\Bigg]$$

$$=e^2\int\frac{\mathrm{d}W^2\,\rho(W^2)}{(p^2-W^2)(p'^2-W^2)}\Gamma^{\mathrm{T}}_{\nu_2}(p',p|\,W), \tag{8}$$

where $\Gamma_{\nu_2}(p',p|\,W)$ denotes the $O(e^2)$ PT expression for Γ_ν, for mesons of mass W. This is an important equation. Equation (3) stipulates that the full longitudinal three-point amplitude is a weighted sum over the lowest-order PT longitudinal amplitude. Analogously, (8) gives the full transverse three-point amplitude as a weighted sum over the lowest-order transverse amplitude in PT. Note that replacing $\rho(W^2)$ in (8) with $\delta(W^2-m^2)$ trivially reproduces $O(e^2)$ PT for all possible values of p and p', in contrast to the vertex used by King. It is also worth noting that the factor (p'^2-p^2) in the denominator of G^{T}_ν can be removed (if so desired) by performing the t integral, which gives rise to an identical factor in the numerator. Further, the expression for G^{T}_ν given in (8) satisfies the differential Ward identity, which can be stated as†

$$G^{\mathrm{T}}_\mu(p,p)=0.$$

3. The refined gauge technique equation

Truncating the Schwinger–Dyson equation for the propagator so that terms starting at $O(e^6)$ or higher are neglected results in

$$Z_\phi^{-1}=(p^2-m_0^2)\Delta(p)-\mathrm{i}e^2\int\bar{\mathrm{d}}^4q\,(2p-q)_\nu D_0^{\nu\mu}(q)G_\mu(p-q,p)$$

$$+e^4\int\bar{\mathrm{d}}^4t\,\bar{\mathrm{d}}^4t'\,g_{\mu\nu}D_0^{\mu\alpha}(t')D_0^{\nu\beta}(t)G^{\mathrm{L}}_{\alpha\beta}(p-t-t',\,t',\,t;\,p), \tag{9}$$

or equivalently

$$0=\int\frac{\mathrm{d}W^2\,\rho(W^2)}{(p^2-W^2)}[W^2-m_0^2+\Pi(p^2,\,W^2)]. \tag{10}$$

It is important to realise that Π in (10) is not the quantity that would result from an $O(e^4)$ PT calculation of the meson self energy $\hat{\Pi}$. For instance, the diagram in figure 1 contributes to $\hat{\Pi}$ but not to Π. However, to $O(e^2)$, $\Pi=\hat{\Pi}$. Following the GT prescription, we take the imaginary part of (10) and obtain

$$\pi\rho(p^2)[p^2-m_0^2+\mathrm{Re}\,\Pi(p^2,p^2)]=\int\frac{\mathrm{d}W^2\,\rho(W^2)}{(p^2-W^2)}\mathrm{Im}\,\Pi(p^2,\,W^2), \tag{11}$$

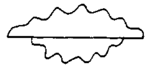


Figure 1. A diagram that contributes to the fourth-order perturbation theory expression for the meson self energy, but not to the gauge technique quantity Π.

† This version of the identity is obtained from the usual statement, $\partial_\mu\Delta(p)=-G_\mu(p,p)$, by using the Lehmann spectral representation of $\Delta(p)$ and arriving at $\partial_\mu\Delta(p)=-G^{\mathrm{L}}_\mu(p,p)$.

where, in terms of Feynman diagrams, π is given in figure 2. Nakanishi's method (Nakanishi 1971) and dimensional regularisation will be employed to calculate $\operatorname{Im}\Pi$.

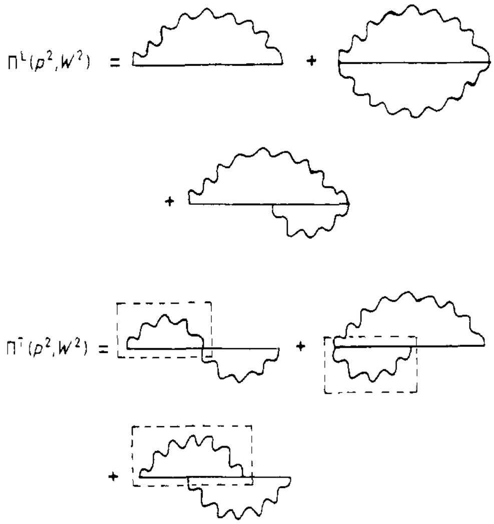


Figure 2. The Feynman diagram expansion of Π. $\Pi^{L}(\Pi^{T})$ denotes the contribution to Π arising from G^{L}_{μ} and $G^{L}_{\mu\nu}(G^{T}_{\mu})$, $\Pi = \Pi^{L} + \Pi^{T}$. Transverse vertices are enclosed in broken line boxes.

For simplicity the Fermi gauge ($a = 1$) will be used from here on. Relegating mathematical details to the appendix, the calculation yields

$$
\begin{aligned}
\operatorname{Im}\Pi(p^2, W^2) &= \pi\eta\theta(p^2 - W^2)\left[\!\!\left[2(W^4 - p^4) + \eta\left\{ 6(p^4 - W^4)\left[\lim_{l\to 2}\frac{1}{(2-l)} - \gamma - \ln\left(\frac{W^2}{4\pi m^2}\right)\right]\right.\right.\right. \\
&\quad - 2(p^2 - W^2)(p^2 + 7W^2) - 4p^2(p^2 + 3W^2)\ln(p^2/W^2) \\
&\quad + 2\left(\frac{p^2 - W^2}{p^2}\right)(p^4 + 14p^2W^2 + W^4)\ln\left(\frac{p^2 - W^2}{W^2}\right) - (p^2 + 3W^2)^2 \\
&\quad \left.\left.\left.\times\left[2\ln\left(\frac{p^2}{W^2}\right)\ln\left(\frac{p^2 - W^2}{W^2}\right) + 3f\left(\frac{p^2}{W^2}\right)\right]\right\}\right]\!\!\right]/p^2 + \theta(p^2 - 9W^2)I,
\end{aligned} \tag{12}
$$

where throughout $\eta = e^2/16\pi^2$. In (12), f denotes the Spence function or dilogarithm (Abramowitz and Stegun 1965) which is defined by

$$
f(x) \equiv \int_1^x \frac{\mathrm{d}t \ln t}{(1-t)},
$$

and γ denotes Euler's constant. The $\theta(p^2 - 9W^2)I$ term originates from the last diagram associated with (A7) and is given by

$$
\begin{aligned}
I = \frac{2\pi\eta^2}{p^2}\int_0^{(p-W)^2} \mathrm{d}q^2 &\left(2pR + \frac{(2p^2 + 2W^2 - q^2)(3p^2 + W^2 - 2q^2)}{(q^2 + W^2 - p^2)} \right. \\
&\left. \times \ln\left|\frac{4pR + (q^2 + W^2 - p^2)}{4pR - (q^2 + W^2 - p^2)}\right| \right),
\end{aligned} \tag{13}
$$

where $p=(p^2)^{1/2}$ is a scalar, R is given by

$$4pR=\{[2(p^2-W^2)-q^2][2(p^2+W^2)-q^2]^{-1}\}^{1/2}\Delta(p^2,\,W^2,\,q^2),$$

and Δ here denotes the triangle function

$$\Delta(p^2,\,W^2,\,q^2)=[(q^2-p^2-W^2)^2-4p^2W^2]^{1/2}.$$

Having calculated $\operatorname{Im}\Pi$, the only remaining quantity to be determined in (11) is $\operatorname{Re}\Pi(p^2,p^2)$. As we shall presently show it will suffice to calculate $\operatorname{Re}\Pi_2(p^2,p^2)$, which results in

$$e^2\operatorname{Re}\Pi_2(p^2,m^2)=-\eta\left[(2p^2+m^2)\left(\lim_{l\to 2}\frac{1}{(2-l)}-\gamma+\ln(4\pi)\right)\right.$$
$$\left.+4p^2+3m^2-2\frac{(p^4-m^4)}{p^2}\ln\left(\frac{p^2-m^2}{m^2}\right)\right]. \tag{14}$$

Equation (11) must be 'renormalised' before it can be considered as an equation for ρ. Because (9) is only valid to $O(e^4)$ in PT, any infinities of $O(e^6)$ or higher will not be cancelled internally. Hence the renormalisation procedure is only expected to work to $O(e^4)$ and any other divergences in (11) can be legitimately removed by hand. Expanding the LHS of (11) to $O(e^4)$ and noting that

$$\Pi_4(m^2,m^2)+\delta m_4^2=0$$

(because the GT can reproduce PT), the only divergence that must be considered is the one present in $\operatorname{Re}\Pi_2(p^2,p^2)-\operatorname{Re}\Pi_2(m^2,m^2)$. Hence the correct form of the LHS of (11) is

$$\pi\rho(p^2)(p^2-m^2)(1-7\eta)+6\pi\eta^2\frac{(p^2+m^2)}{p^2}\left(\lim_{l\to 2}\frac{1}{(2-l)}-\gamma+\ln(4\pi)\right). \tag{15}$$

The RHS of (11) has the form

$$\int\frac{\mathrm{d}W^2\,\rho(W^2)}{(p^2-W^2)}\left\{6\pi\eta^2\frac{(p^4-W^4)}{p^2}\left[\lim_{l\to 2}\frac{1}{(2-l)}-\gamma+\ln\left(\frac{W^2}{4\pi m^2}\right)+C\right]+\text{finite terms}\right\},$$

where the finite constant C has been included to take account of the ambiguity in the infinite part of the expression. The only legitimate divergence is found by replacing $\rho(W^2)$ by $\delta(W^2-m^2)$ leading to

$$\int\frac{\mathrm{d}W^2\,\rho(W^2)}{(p^2-W^2)}\left\{6\pi\eta^2\frac{(p^4-W^4)}{p^2}\left[C-\ln\left(\frac{W^2}{m^2}\right)\right]+\text{finite terms}\right\}$$
$$+6\pi\eta^2\frac{(p^2+m^2)}{p^2}\left(\lim_{l\to 2}\frac{1}{(2-l)}-\gamma+\ln(4\pi)\right). \tag{16}$$

Combining (15) and (16) yields the finite, linear equation for ρ,

$$(p^2-m^2)(1-7\eta)\rho(p^2)=\eta\int_{m^2}^{p^2}\frac{\mathrm{d}W^2\,\rho(W^2)}{p^2(p^2-W^2)}\left[\!\!\left[2(W^4-p^4)+\eta\left\{6(p^4-W^4)\right.\right.\right.$$
$$\times[C-\ln(W^2/m^2)]-2(p^2-W^2)(p^2+7W^2)-4p^2(p^2+3W^2)\ln(p^2/W^2)$$
$$-(p^2+3W^2)^2\left[2\ln\left(\frac{p^2}{W^2}\right)\ln\left(\frac{p^2-W^2}{W^2}\right)+3f\left(\frac{p^2}{W^2}\right)\right]$$

$$+2\frac{(p^2-W^2)}{p^2}(p^4+14p^2W^2+W^4)\ln\left(\frac{p^2-W^2}{W^2}\right)\bigg\}\bigg]\bigg]$$

$$+\theta(p^2-9m^2)\int_{m^2}^{p^2/9}\frac{dW^2\,\rho(W^2)I}{\pi(p^2-W^2)}. \tag{17}$$

The value of C will be determined by demanding that (17) reproduce the well known IR behaviour of ρ. Recall that the zeroth-order GT is valid in the IR regime.

4. Asymptotic behaviours of ρ

Due to the algebraic complexity of (17) we shall only analyse its behaviour in asymptotic regimes. In the IR limit ($p^2\to m^2$), (17) reduces to

$$(p^2-m^2)(1-7\eta)\rho(p^2)=-4\eta\int_{m^2}^{p^2}\mathrm{d}W^2\,\rho(W^2)[1-(3C+4)\eta]. \tag{18}$$

with $C=1$, (18) becomes the Delbourgo–West equation (Delbourgo and West 1977b) which gives rise to the standard expression

$$\rho(p^2)\to\text{constant}(p^2-m^2)^{-1-4\eta},\qquad p^2\to m^2. \tag{19}$$

In terms of the dimensionless variables

$$x=p^2/m^2,\qquad y=W^2/m^2,\qquad \phi(x)=m^2\rho(p^2),$$

(17) can be approximated in the UV by

$$(1-7\eta)x^2\phi(x)=\eta\int_1^x\mathrm{d}y\,\phi(y)\bigg[-2(x+y)+\eta\bigg(-6(x+y)\ln y+4(x-2y)$$

$$+2\frac{(x^2+14xy+y^2)}{x}\ln\left(\frac{x-y}{y}\right)-\frac{1}{(x-y)}\bigg\{4x(x+3y)\ln\left(\frac{x}{y}\right)+(x+3y)^2$$

$$\times\left[2\ln\left(\frac{x}{y}\right)\ln\left(\frac{x-y}{y}\right)+3f\left(\frac{x}{y}\right)\right]\bigg\}\bigg)\bigg]+x\int_1^{x/9}\mathrm{d}y\frac{\phi(y)J(x,y)}{(x-y)}, \tag{20}$$

where $\pi m^2J(x,y)=I(p^2,W^2)$ and I is given in (13). Adopting the UV ansatz

$$\phi(x)\sim x^a(1+b\ln x), \tag{21}$$

and knowing that each integral on the RHS of (20) is dominated by the upper end point, and that

$$a=-1+k\eta,$$

means that every integral in (20) (except the one involving J) can be done analytically. In every case except one, the $x^a(\ln x)^n$ behaviour, with $n=2,3,4$, *exactly* cancels internally, resulting in terms of the form given in (21). The single integral that does not conform is the one involving the $(x+y)\ln y$ term; it gives rise to a $x^a(\ln x)^2$ term. The $(x+y)\ln y$ term is present because the replacement $e^2\to e^2(m^2)^{2-l}$ was made in the $2l$-dimensional space–time calculations (12) and (14). In other words, m^2 was the parameter introduced to maintain a dimensionless coupling constant in $2l$ dimensions. If W^2 had been used instead, the $(x+y)\ln y$ term would not arise, and (21) would lead to a self-consistent equation†. Further, this replacement would not affect the IR results since in that limit $W^2 \to m^2$ and the term vanishes. Because the GT has been used throughout this article, and the GT deals with mesons of mass W, it is reasonable to use W^2 rather than m^2 in the redefinition $e^2 \to e^2(W^2)^{2-l}$ for arbitrary dimensions. In this circumstance the $(x+y)\ln y$ term in (20) can be dropped.

From here on, we confine the analysis to the case of small η. This restriction, together with the knowledge that b and k need only be determined to leading order, implies that only those integrands in (20) that do not contain $(x-y)^{-1}$ need be considered. If an integrand contains $(x-y)^{-1}$, the result will not contain an $(a+1)^{-n}$ term and hence the contribution of that integral will be suppressed by a positive integral power of η. This explains why the integral involving J can be neglected.

With the modifications described (20) is transformed into a self-consistent equation for b and k. Solving the equation yields the values

$$k = -2 + \mathrm{O}(\eta), \qquad b = -6\eta^2 + \mathrm{O}(\eta^3), \tag{22}$$

and hence the UV behaviour of ϕ is given by

$$\phi(x) \sim x^{-1-2\eta}(1 - 6\eta^2 \ln x), \qquad x \to \infty. \tag{23}$$

Compare this with the IR behaviour, (19)

$$\phi(x) \sim (x-1)^{-1-4\eta}, \qquad x \to 1.$$

Equation (23) implies that the dominant UV behaviour is given by a power law and is obtainable by using the zeroth-order GT. The inclusion of transverse amplitudes only yields subdominant corrections to the power law behaviour.

5. Conclusion

We have described a new algorithm for introducing transverse corrections into the GT. The algorithm has several advantages over previously advocated schemes.

(1) It is intrinsically non-perturbative.

(2) It yields an expression (which exactly agrees with $\mathrm{O}(e^2)$ PT everywhere) for the full three-point transverse amplitude and this expression has the same structure as the GT expression for the full three-point longitudinal amplitude.

(3) In principle it yields a propagator which is exact to $\mathrm{O}(e^4)$ and is valid in all momentum regimes.

(4) What is more, it is usable.

The method gives the exact IR and correct UV behaviours of the spectral function. However, (8) only represents an approximation to the true G^{T}_{μ} because it was derived by using only the longitudinal amplitudes G^{L}_{μ} and $G^{\mathrm{L}}_{\mu\nu}$ in a truncated Schwinger–Dyson equation. Transverse vertices must be considered in intermediate momentum regimes if the GT is to provide gauge covariant solutions everywhere.

The current work could be used as the basis for several investigations. One possibility is to use our G_{μ} $(=(3)+(8))$ in a non-perturbative calculation of the charge form factor. Note that our G_{μ} contains the contribution of the $\mathrm{O}(e^2)$ triangle diagram that is usually used in the calculation. Another possibility is to perform a numerical

† A numerical analysis of the integral involving J has revealed that only terms of the form given in (21) are present.

analysis of (17) and obtain a refined approximation to ρ for all p^2. This is a worthwhile topic since it is only in subasymptotic regimes where the effects of transverse contributions become evident.

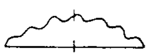
For QED the zeroth-order GT only respects gauge covariance in the IR and UV limits (Delbourgo *et al* 1981). The most important application of the new algorithm will be to determine whether the transverse corrections introduced via the method are sufficient to ensure the gauge covariance of the GT solutions for S and G_μ. This question is currently being investigated and the results will be reported elsewhere.

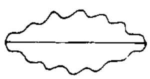
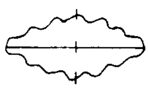
Acknowledgment

It is a pleasure to thank Professor R Delbourgo for his continual encouragement and assistance throughout the course of this work.

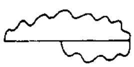
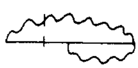
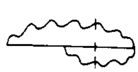
Appendix

It is the purpose of this appendix to present a calculation of Im Π. With the notation that a short vertical bar through a propagator denotes the imaginary part of that propagator, we have (from figure 2)

Im  =  $= -2\pi\eta(p^4 - W^4)\theta(p^2 - W^2)/p^2$, (A1)

Im  = 

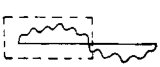
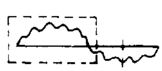
$$= 4\pi\eta^2\theta(p^2 - W^2)[p^4 - W^4 - 2p^2W^2\ln(p^2/W^2)]/p^2, \qquad \text{(A2)}$$

Im  =  + 

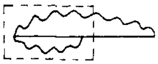
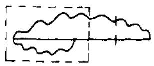
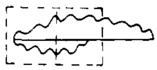
$$= \pi\eta^2\theta(p^2 - W^2)\left\{6(p^2 - W^2)(p^2 + 3W^2)\left[\lim_{l\to 2}\frac{1}{(2-l)} - \gamma - \ln\left(\frac{W^2}{4\pi m^2}\right)\right]\right.$$
$$\left. + 3(p^2 - W^2)(3p^2 + 19W^2) - 2p^2(3p^2 + 2W^2)\ln(p^2/W^2)\right\}/(4p^2), \qquad \text{(A3)}$$

and combining these expressions yields

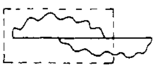
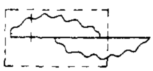
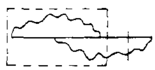
$$\operatorname{Im}\Pi^{\mathrm{L}}(p^2, W^2) = \pi\eta\theta(p^2 - W^2)\left[\!\!\left[2(W^4 - p^4)\right.\right.$$
$$+ \eta\left\{6(p^2 - W^2)(p^2 + 3W^2)\left[\lim_{l\to 2}\frac{1}{(2-l)} - \gamma - \ln\left(\frac{W^2}{4\pi m^2}\right)\right]\right.$$
$$\left.\left.\left. + (p^2 - W^2)(25p^2 + 73W^2) - 6p^2(p^2 + 6W^2)\ln(p^2/W^2)\right\}/4\right]\!\!\right]/p^2. \qquad \text{(A4)}$$

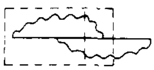
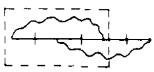
Im  =  + 

$$
\begin{aligned}
&= \pi\eta^2\theta(p^2-W^2)\bigg\{6p^2(p^2-W^2)\bigg[\lim_{l\to 2}\frac{1}{(2-l)}-\gamma-\ln\bigg(\frac{W^2}{4\pi m^2}\bigg)\bigg] \\
&\quad +2(p^2-W^2)(4p^2-W^2)+(p^2-W^2)^2\frac{(p^2+W^2)}{p^2} \\
&\quad -p^2(3p^2+W^2)\ln\bigg(\frac{p^2}{W^2}\bigg)-\frac{(p^2-W^2)^3}{p^4}(3p^2+W^2)\ln\bigg(\frac{p^2-W^2}{W^2}\bigg)\bigg\}/p^2,
\end{aligned} \tag{A5}
$$

Im  =  + 

$$
\begin{aligned}
&= -\pi\eta^2\theta(p^2-W^2)\bigg\{6(p^2-W^2)^2\bigg[\lim_{l\to 2}\frac{1}{(2-l)}-\gamma-\ln\bigg(\frac{W^2}{4\pi m^2}\bigg)\bigg] \\
&\quad +(p^2-W^2)(21p^2-19W^2)-2p^2(3p^2-2W^2)\ln(p^2/W^2)\bigg\}/(4p^2),
\end{aligned} \tag{A6}
$$

Im  =  + 

\+  + 

$$
\begin{aligned}
&= \pi\eta^2\theta(p^2-W^2)\bigg\{(W^2-p^2)(13p^2+33W^2)+\frac{(p^2-W^2)^3}{p^2} \\
&\quad -p^2(p^2+W^2)\ln\bigg(\frac{p^2}{W^2}\bigg)-(p^2+3W^2)^2\bigg[2\ln\bigg(\frac{p^2}{W^2}\bigg)\ln\bigg(\frac{p^2-W^2}{W^2}\bigg) \\
&\quad +3f\bigg(\frac{p^2}{W^2}\bigg)\bigg]+(5p^6+23p^4W^2+3p^2W^4+W^6)\frac{(p^2-W^2)}{p^4} \\
&\quad \times\ln\bigg(\frac{p^2-W^2}{W^2}\bigg)\bigg\}/p^2+\theta(p^2-9W^2)I,
\end{aligned} \tag{A7}
$$

and combining these expressions results in

$$
\begin{aligned}
\operatorname{Im}\Pi^{\mathrm{T}}(p^2,W^2) &= \pi\eta^2\theta(p^2-W^2)\bigg\{6(p^2-W^2)(3p^2+W^2) \\
&\quad \times\bigg[\lim_{l\to 2}\frac{1}{(2-l)}-\gamma-\ln\bigg(\frac{W^2}{4\pi m^2}\bigg)\bigg]+(W^2-p^2)(33p^2+129W^2) \\
&\quad -2p^2(5p^2+6W^2)\ln\bigg(\frac{p^2}{W^2}\bigg)-4(p^2+3W^2)^2\bigg[2\ln\bigg(\frac{p^2}{W^2}\bigg)\ln\bigg(\frac{p^2-W^2}{W^2}\bigg) \\
&\quad +3f\bigg(\frac{p^2}{W^2}\bigg)\bigg]+8(p^2-W^2)\frac{(p^4+14p^2W^2+W^4)}{p^2}\ln\bigg(\frac{p^2-W^2}{W^2}\bigg)\bigg\}/(4p^2) \\
&\quad +\theta(p^2-9W^2)I.
\end{aligned} \tag{A8}
$$

I and f are given in the text.

References

Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
Ball J S, Horn D and Zachariasen F 1978 *Nucl. Phys.* B **132** 509
Delbourgo R 1977 *J. Phys. A: Math. Gen.* **11** 2057
—— 1979 *Nuovo Cimento* **49A** 484
Delbourgo R, Keck B W and Parker C N 1981 *J. Phys. A: Math. Gen.* **14** 921
Delbourgo R and Thompson G 1982 *J. Phys. G: Nucl. Phys.* **8** L185
Delbourgo R and West P 1977a *J. Phys. A: Math. Gen.* **10** 1049
—— 1977b *Phys. Lett.* **72B** 96
Gardner E J 1981 *J. Phys. G: Nucl. Phys.* **7** L269
Gorkov L P 1956 *Sov. Phys.–JETP* **3** 762
Hagen C 1963 *Phys. Rev.* **130** 813
Khare A and Kumar S 1978 *Phys. Lett.* **78B** 94
Kibble T 1968 *Phys. Rev.* **174** 1882
King J E 1983 *Phys. Rev.* D **27** 1821
Nakanishi N 1971 *Graph Theory and Feynman Integrals* (New York: Gordon and Breach)
Salam A 1963 *Phys. Rev.* **130** 1287
Slim H A 1981a *Nucl. Phys.* B **177** 172
—— 1981b *PhD Thesis, University of Groningen* (unpublished)